Construction of the Lebesgue Integral

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Throughout this chapter we focus on spaces $(\Omega, \mathcal{F}, \mu)$ and $(\mathbb{R}, \mathcal{R}, \lambda)$. For the generalization to a $\mathbb{R}^k$ space you are invited to have a look on Billingsley (Chapter 2, 12-13). Aim of this lecture is to give a meaning to an integral against the Lebesgue measure and point out the differences between and the Lebesgue and Riemann integral.

1 Null sets

Note that this part is not a preliminary to the part devoted to the Lebesgue integral, so you may safely skip it. It is included mostly to highlight the power of Lebesgue measure theory and make a notion of something being held "almost surely" more clear. Second application of null sets is that often, in order to prove that some set A is measurable, we may try to find some "easier" set B, which differs from A only on a null set, and then represent B as a countable union of Borel sets.

Definition 1 (Null set)**.** *Set A is called a null set if $\forall \varepsilon > 0 \quad \exists \{U_k\}$ s.t.:*

(i) $\sum_{k=1}^{\infty} \lambda(U_k) < \varepsilon$

(ii) $A \subset \cup_{k=1}^{\infty} U_k$,

From this definition we may infer that empty set, any countable set of real numbers and any countable union of null sets are again null sets. Famous example of a null set which is not countable is the Cantor set[1]:

Example 1 (Cantor Set)**.** *The Cantor set $\mathcal{C}$ is created by iteratively deleting the* ***open*** *middle third from a unit interval $[0, 1]$. On the first step $\mathcal{C}$ contains $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, on the second — $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ and so on. Whatever remains at the limit is a Cantor set. Then, we may compute its measure by computing the following difference:*

$$\lambda(\mathcal{C}) = \lambda([0, 1]) - \lambda\left(\cup_{n=1}^{\infty} \mathcal{C}_n\right),$$

[1] Illustrations are from en.wikipedia.org

where $\mathcal{C}_n$ is a part removed on n-th iteration. Then

$$\lambda(\mathcal{C}) = 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = 1 - \left(\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots\right) = 1 - \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 0$$

Figure 1: Cantor set (first 7 iterations)

Somewhat more disturbing example is a Smith-Volterra-Cantor set. It is constructed in almost the same way and is topologically equal to a Cantor set, nowhere dense[2] and not countable, yet having a positive measure:

Example 2 (Smith-Volterra-Cantor set)**.** *For a Smith-Volterra-Cantor set we proceed in exactly the same manner as in previous example, but instead of removing* $\frac{1}{3}$ *of each remaining interval, we remove* $\frac{1}{4^n}$ *on each step. Then, on the first step we have* $[0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$*, on the second —* $[0, \frac{5}{32}] \cup [\frac{7}{32}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{25}{32}] \cup [\frac{27}{32}, 1]$ *etc. By the same logic, the measure of the resulting set is equal to*

$$1 - \sum_{n=0}^{\infty} \frac{2^n}{2^{2n+2}} = 1 - \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) = \frac{1}{2}$$

Figure 2: Smith-Cantor-Volterra set (first 5 iterations)

[2]a subset A of a topological space X is called dense (in X) if every point $x \in X$ either belongs to A or is a limit point of A; that is, the closure of A is constituting the whole set X

2 Lebesgue integral

The objective of this part is to give a meaning to the following integral:

$$\int f d\mu = \int_\Omega f(\omega) d\mu(\omega) = \int_\Omega f(\omega)\mu(d\omega) \tag{1}$$

Of course, to ensure the existence of (1), we need to restrict ourselves to the class of **measurable** functions.

Definition 2 (Measurable function)**.** *Function $f : \Omega \to \mathbb{R}$ is called $\mathcal{F}$-measurable, if $\forall A \subset \mathbb{R} \quad f^{-1}(A) = \{\omega \in \Omega : f(\omega) \subset A\} \subset \mathcal{F}$.*

In probabilistic context, a real measurable function is called a **random variable**. Important result is that the class of the measurable functions is closed under the limiting procedures:

Theorem 1. *Suppose that $f_1, f_2, \ldots$ are real $\mathcal{F}$-measurable functions. Then:*

(i) $\sup_n f_n, \inf_n f_n, \limsup_n f_n$ *and* $\liminf_n f_n$ *are also $\mathcal{F}$-measurable.*

(ii) If $\lim_n f_n$ exists everywhere, then it is $\mathcal{F}$-measurable.

(iii) The ω-set, where $\{f_n(\omega)\}$ converges, lies in $\mathcal{F}$.

(iv) If f is $\mathcal{F}$-measurable, the ω-set, where $f_n(\omega) \to f(\omega)$, lies in $\mathcal{F}$.

Proof. $\{\sup_n f_n \leq x\} = \cap_n \{f_n \leq x\} \in \mathcal{F}$ even for $x = \infty$ or $x = -\infty$, and so $\sup_n f_n$ is measurable. Measurability of $\inf_n f_n$ is shown in the same way. Hence, $\limsup_n f_n = \inf_n \sup_{k \geq n} f_k$ and $\liminf_n f_n = \sup_n \inf_{k \geq n} f_k$ are measurable. Then, if $\lim_n f_n$ exists, it coincides with these last two equations and is measurable. Finally, in (iii) is the set where $\limsup_n f_n = \liminf_n f_n$, and in (iv) we deal with the set where this common value is $f(\omega)$. $\square$

2.1 Definition of an integral for a simple function

Before constructing an integral (1) of a general form, let us first start with defining a **simple** function.

Definition 3 (Simple function)**.** *$f : \Omega \to \mathbb{R}$ is called simple if it can be represented as*

$$f = \sum_i x_i \mathbb{1}_{A_i}, \tag{2}$$

where A_i form a finite decomposition of Ω.

Note that f is $\mathcal{F}$-measurable if each A_i lies in $\mathcal{F}$. In this case simple function is an analogous definition of a **discrete random variable**. For functions (2) the integral against the measure μ is defined as:

I. If f is a non-negative simple function $f = \sum_i x_i \mathbb{1}_{A_i}$, then

$$\int f d\mu = \sum_i x_i \mu(A_i) \tag{3}$$

One of the most famous examples to mention here is the Dirichlet function:

Example 3 (Dirichlet function)**.**

$$f(x) = \mathbb{1}_{x \in \mathbb{Q}}, \quad x \in [0,1]$$

Note that this function is not integrable in Riemann sense! However, by (3)*, we simply have:*

$$\int f d\lambda = 1 \cdot \lambda(\mathbb{Q}) + 0 \cdot \lambda([0,1] \backslash \mathbb{Q}) = 0,$$

since the set $\mathbb{Q}$ *is countable (hence, it has a measure* 0*).*

This result can be translated to probabilistic terms as "if we randomly draw a real number from a unit interval, the probability of picking up a rational number is zero". The integral itself gives an expected value of a random variable X, equal to 1 if we picked up a ration number and zero otherwise.

2.2 Definition of an integral for a general function

Good news that for any positive function we may find its approximation by a sequence of simple functions and define the integral for a wider class using definition I. It is stated by the following theorem:

Theorem 2. *If* f *is a real* $\mathcal{F}$*-measurable function, there exists a sequence* $\{f_n\}$ *of simple functions, each* $\mathcal{F}$*-measurable s.t.*

$$0 \leq f_n(\omega) \uparrow f(\omega), \text{ if } f(\omega) \geq 0$$
$$0 \geq f_n(\omega) \downarrow f(\omega), \text{ if } f(\omega) \leq 0$$

Proof. Consider the following sequence of functions:

$$f_n(\omega) = \begin{cases} -n & \text{if } -\infty < f(\omega) \leq -n \\ -(k-1)2^{-n} & \text{if } -k2^{-n} < f(\omega) \leq -(k-1)2^{-n}, 1 \leq k \leq n2^n \\ (k-1)2^{-n} & \text{if } (k-1)2^{-n} \leq f(\omega) \leq k2^{-n}, 1 \leq k \leq n2^n \\ n & \text{if } n \leq f(\omega) < \infty \end{cases}$$

It satisfies the properties. □

Then it makes sense to define an integral for a non-negative function f as a sup of the integrals of the simple functions f_n s.t. $f_n \leq f$. Formally, it can be defined in the following way:

II. If f is a non-negative real, we consider all possible sequences of simple functions $\{f_n^{(i)}, n \geq 1\}$ s.t. $f_n^{(i)} \leq f$, and then

$$\int f d\mu = \sup_i \left\{ \int_\Omega f_n^{(i)} d\mu \right\} \tag{4}$$

This definition may remind you on that of the Riemann integral. The crucial difference here is that the value of the Lebesgue integral does not depend on the value of the function f on null sets. Thus, Lebesgue integral allows us to integrate over a wider class of functions. However, for Riemann-integrable functions values of a Riemann and Lebesgue integral coincide.

Finally, for computing the integral of a general function, we just have to represent it as a linear combination of its positive and negative parts:

III. For a general f, we consider the decomposition $f = f^+ - f^-$, where $f^+(\omega) = f(\omega)\mathbb{1}_{f \geq 0}$, and $f^-(\omega) = -f(\omega)\mathbb{1}_{f \leq 0}$. Note that f^+ and f^- are measurable. Then the general integral is defined as

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu,$$

unless $\int f^+ d\mu = \int f^- d\mu = \infty$, in which case f has no integral

The conventions for computing (1), when some summands in (3) take ∞ as a value, are the following: $\infty \cdot 0 = 0 \cdot \infty, \quad x \cdot \infty = \infty \cdot x = \infty, \quad \infty \cdot \infty = \infty$.

2.3 Properties of Lebesgue integral

(...in development...)

Theorem 3 (Beppo Levi's monotone convergence theorem)**.** *Let $f_n, n \geq 1$ be a sequence of non-decreasing non-negative measurable functions $f_n : \Omega \to \mathbb{R}$. Then*

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega [\lim_{n\to\infty} f_n] d\mu$$

Lemma 1. *Let $f, g : \Omega \to \mathbb{R}$ be two measurable functions. Then*

(i) If $f \leq g$:

$$\int_\Omega f d\mu \leq \int_\Omega g d\mu$$

(ii) $\forall a, b \in \mathbb{R}$ the following holds:

$$\int_\Omega (af + bg) d\mu = a \int_\Omega f d\mu + b \int_\Omega g d\mu$$

Theorem 4 (Fatou's Lemma)**.** *Let f_n be a sequence of positive measurable functions. Then*

$$\int_\Omega [\inf_n \lim f_n] d\mu \leq \inf_n \lim \int_\Omega f_n d\mu$$

Theorem 5. *Suppose that f and g are nonnegative, then*

(i) If $f = 0$ almost everywhere, then $\int f d\mu = 0$.

(ii) If $\mu[\omega : f(\omega) > 0] > 0$, then $\int f d\mu > 0$.

(iii) If $f d\mu < \infty$, then $f < \infty$ almost everywhere.

(iv) If $f \leq g$ almost everywhere, then $\int f d\mu \leq \int g d\mu$.

(v) If $f = g$ almost everywhere, then $\int f d\mu = \int g d\mu$.