M1 MSIAM Fall 2020

Homework #1 Applied Probability

The scan of the homework formated as .pdf *to be uploaded by November 1, 11:59 pm*

Exercise 1

Suppose that $m \in \mathbf{R}$, $n \in \mathbf{N}^*$, and $\{X_i\}_{i=1,\dots,n}$ are independent random variables. We set

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

1^o. Assume that r.v. $X_i - m$ have Cauchy distribution with density

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

What is the distribution of $\bar{X}$? Does $\bar{X}$ have second order moments or first order moments? Compare the tails of the Cauchy distribution with those of $N(0,1)$ distribution (for instance, compute $P(X > 3)$ and $P(N(0,1) > 3)$).

2^o. We suppose that $X_1, \dots, X_n$ are i.i.d. random variables with exponential distribution $\mathcal{E}(\theta)$ with density

$$f(x) = \theta e^{-\theta x} I(x \geq 0).$$

Identify the distribution of $\bar{X}$.

3^o. Let now X_i, $i = 1, \dots, n$, be i.i.d. Poisson random variables with parameter $\lambda > 0$. What is the distribution of $n\bar{X}$? Find two sequences a_n and b_n such that $a_n\bar{X} + b_n$ converge weakly (in distribution) to a random variable with a nondegenerate distribution.

Exercise 2

We consider two independent random variables U and V which follow $N(0,1)$ and we define the following random variable:

$$X = \frac{U}{V}.$$

Show that X follows Cauchy law.

Exercise 3

Let $X_1, \dots, X_n$ be independent random variables following the law $N(0,1)$ and $a_1, \dots, a_n, b_1, \dots, b_n$ some real numbers. Show that $Y = \sum_{i=1}^n a_i X_i$ and $Z = \sum_{i=1}^n b_i X_i$ are independent if and only if $\sum_{i=1}^n a_i b_i = 0$.

Exercise 4

Let X be a standard Gaussian variable. For all $c > 0$, we put

$$X_c = X\left(I(|X| < c) - I(|X| \geq c)\right).$$

1) Write the law of X_c.

2) Compute $\mathrm{Cov}(X, X_c)$ and show that there exist c_0 such that $\mathrm{Cov}(X, X_{c_0}) = 0$.

3) Show that X and X_{c_0} are not independent. Is the vector (X, X_{c_0}) Gaussian?

Exercise 5

Let (X, Y, Z) be a Gaussian vector $\mathcal{N}_3(\mu, \Sigma)$ with the mean vector and the covariance matrix given, respectively, by:

$$\mu = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} , \Sigma = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix} .$$

1) Write the law of X, and the joint law of Y and $2Y + Z$.

2) Write the law of X given Z, and the law of Z given (X, Y).

3) Write the law of X given $2Y + Z$.