Statistical testing of the covariance matrix rank in multidimensional neuronal models

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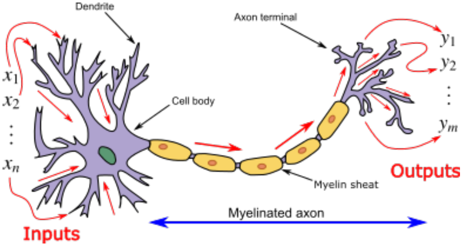
Part I: Motivation

Motivation



Dendrite
Axon terminal
Cell body
Myelin sheat
Myelinated axon
Inputs
Outputs
x_1
x_2
x_n
y_1
y_2
y_m

Example 1: Hodgkin-Huxley model

Conductance-based model of **action potential in neurons**:

$$\begin{cases} I & = C_m \frac{\mathrm{d}V_m}{\mathrm{d}t} + \bar{g}_{\mathrm{K}} n^4 (V_m - V_K) + \bar{g}_{\mathrm{Na}} m^3 h (V_m - V_{Na}) + \bar{g}_l (V_m - V_l) \\ \frac{dn}{dt} & = \alpha_n(V_m)(1-n) - \beta_n(V_m) n \\ \frac{dm}{dt} & = \alpha_m(V_m)(1-m) - \beta_m(V_m) m \\ \frac{dh}{dt} & = \alpha_h(V_m)(1-h) - \beta_h(V_m) h \end{cases}$$

- I — membrane potential
- n, m, h — quantities between 0 and 1 that are associated with potassium channel activation, sodium channel activation, and sodium channel inactivation.

References: Hodgkin and Huxley (1952) — *1963 Nobel Prize in Physiology or Medicine*,
Modifications: Fitzhugh (1961), Morris and Lecar (1981)

Example 2: Jansen and Rit Neural Mass model

Convolution-based model of a **neuronal population with excitatory and enhibitory subpopulations**:

$$\begin{cases} dQ(t) = \nabla_P H(Q, P)\,dt, \\ dP(t) = (-\nabla H(Q, P) - 2\Gamma P + G(t, Q))\,dt + \Sigma(t)\,dW_t, \end{cases}$$

- $Q = (X_0, X_1, X_2) \in \mathbb{R}^3$, $P = (X_3, X_4, X_5) \in \mathbb{R}^3$
- $\Gamma = \text{diag}[a, a, b] \in \mathbb{R}^{3\times 3}$ is a damping part,
- $\Sigma(t) = \text{diag}[\sigma_3(t), \sigma_4(t), \sigma_5(t)] \in \mathbb{R}^{3\times 3}$ is a diffusion part,
- $G(t, Q)$ is a nonlinear displacement term.
- Diffusion components are of **different order** (e.g. $\sigma_3(t), \sigma_5(t) \ll \sigma_4(t)$)!

References: Jansen and Rit (1995), Ableidinger et al. (2017), Buckwar et al. (2)

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Example 2: Jansen and Rit Neural Mass model

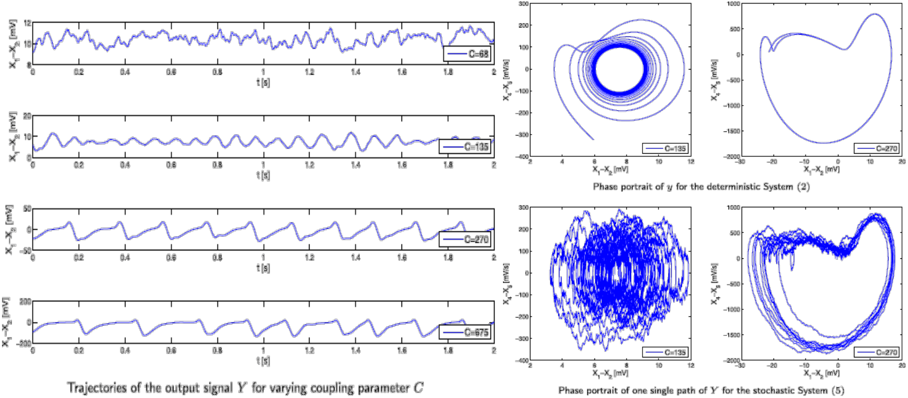

Figure: Source: Ableidinger et al. (2017)

Example 3: Diffusion approximation of a Hawkes process

Hawkes process (point process with memory), describing the action potentials in a population of neurons, can be approximated by a stochastic diffusion:

$$dZ_t = A(Z_t)dt + B(Z_t)dW_t.$$

$$A(z) = \begin{pmatrix} -\nu_1 z^1 + z^2 \\ -\nu_1 z^2 + z^3 \\ \vdots \\ -\nu_1 z^{\eta_1+1} + c_1 f_2(z^{\eta_1+2}) \\ -\nu_2 z^{\eta_1+2} + z^{\eta_1+3} \\ \vdots \\ -\nu_n z^{\kappa} + c_n f_1(z^1) \end{pmatrix}, \quad B(z) = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & \frac{c_1}{\sqrt{p_2}}\sqrt{f_2(z^{\eta_1+2})} \\ 0 & 0 \\ \vdots & \vdots \\ \frac{c_n}{\sqrt{p_1}}\sqrt{f_1(z^1)} & 0 \end{pmatrix},$$

Reference: Ditlevsen and Löcherbach (2017), Chevallier (2017)

Example 3: Diffusion approximation of a Hawkes process

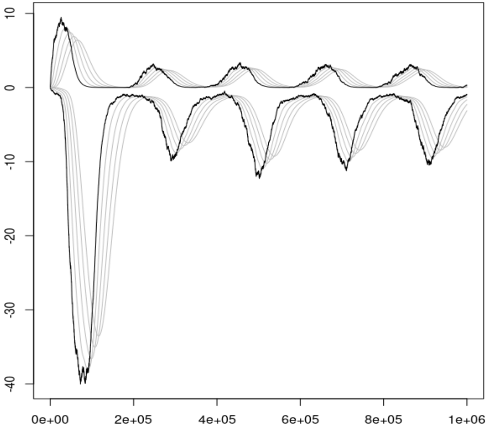

Figure: Diffusion approximation of Hawkes process describing inhibitory and excitatory neuron population (20 neurons in each population)

Where to put noise?

Main challenges:

- Highly non-linear systems
- Computational cost
- Measurements inaccuracy
- Diffusion coefficients of different orders

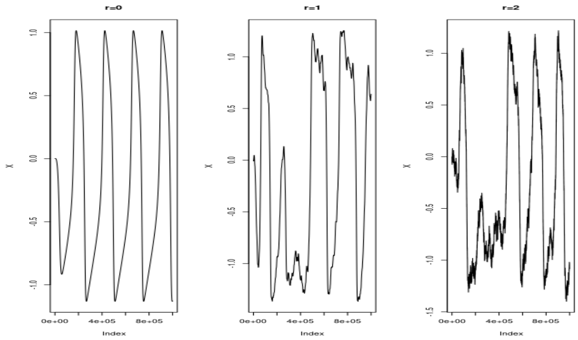

Figure: Membrane potential simulated wiht a FitzHugh-Nagumo model: deterministic, noisy channels, elliptic system

References: see Tuckwell (2005) for general overview of stochastic neuronal models

Part II: Problem statement and preliminaries

Formalization

Given:

Discrete observations X_i of the d-dimensional process

$$dX_t = A_t dt + B_t dW_t,\ t \in [0, T], \tag{1}$$

$A_t \in \mathbb{R}^d$, $B_t \in \mathbb{R}^{d\times q}$.

Goal:

Propose a test

$$H_0 : rank(\Sigma) = r_0$$
$$H_1 : rank(\Sigma) \neq r_0,$$

where $\Sigma = B_t B_t^T$. If B_t is not constant, we search $\sup_t \text{rank}(\Sigma)$ instead.

Other applications

- Financial mathematics:
  - Volatility rank in multi-assets portfolio
- Sensitivity analysis:
  - Number of influential inputs
- Number of components in noisy data
  - Estimate rank (or biggest eigenvalues) of D, given only observations

$$D + E,$$

where E — unknown matrix of noise.

References: Konstantinides and Yao (1988), Zarowski (1998), Kritchman and Nadler (2008), Iooss and Lemaître (2015)

Applications: signal processing, chemometrics, genomics.

What do we want to study?

Given a d-dimensional process $i = 1, \dots, N$, we study the matrices of increments:

$$\bar{s}_i := \begin{pmatrix} \frac{X^{(1)}_{id+1} - X^{(1)}_{id}}{\sqrt{\Delta}} & \frac{X^{(1)}_{id+2} - X^{(1)}_{id+1}}{\sqrt{\Delta}} & \dots & \frac{X^{(1)}_{id+d} - X^{(1)}_{id+d-1}}{\sqrt{\Delta}} \\ \frac{X^{(2)}_{id+1} - X^{(2)}_{id}}{\sqrt{\Delta}} & \frac{X^{(2)}_{id+2} - X^{(2)}_{id+1}}{\sqrt{\Delta}} & \dots & \frac{X^{(2)}_{id+2d} - X^{(2)}_{id+d-1}}{\sqrt{\Delta}} \\ \dots & \dots & \dots & \dots \\ \frac{X^{(d)}_{id+1} - X^{(d)}_{id}}{\sqrt{\Delta}} & \frac{X^{(d)}_{id+2} - X^{(d)}_{id+1}}{\sqrt{\Delta}} & \dots & \frac{X^{(d)}_{id+d} - X^{(d)}_{id+d-1}}{\sqrt{\Delta}} \end{pmatrix} \tag{2}$$

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What is the determinant?

For a 3-dimensional system the determinant

$$\det\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} =$$

$$e_{11}\det\begin{pmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{pmatrix} - e_{12}\det\begin{pmatrix} e_{21} & e_{23} \\ e_{31} & e_{33} \end{pmatrix} + e_{13}\det\begin{pmatrix} e_{21} & e_{22} \\ e_{31} & e_{32} \end{pmatrix} =$$

$$e_{11}e_{22}e_{33} - e_{11}e_{32}e_{23} - e_{12}e_{21}e_{33} + e_{12}e_{31}e_{23} + e_{13}e_{21}e_{32} - e_{13}e_{31}e_{22}$$

Straightforward approach: Jacod et al. (2008)

Main idea: check if the determinant of the submatrix of a given rank is "small enough"

- Compute matrices of increments (2) (squares of it)
- Define a converging sequence $\{\rho_N\}$ of "thresholds", s.t.:

$$\rho_N \to 0, \ \rho_N \sqrt{N} \to \infty$$

- Define by $\mathcal{A}_r$ class of all subsets of $\{1, \ldots d\}$ with r elements. Define $\det_K \Sigma$ the determinant of $r \times r$ submatrix of $\Sigma^{kl}, \ k, l \in K$. Finally, define:

$$\det(r; \Sigma) = \sum_{K \in \mathcal{A}_r} \det_K \Sigma \tag{3}$$

- $\widehat{R}_{T,\Delta} = \inf \left\{ r \in \{0, \ldots, d-1\} : \frac{1}{\Delta(r+1)!} \sum_{i=1}^{N-r+1} \det(r+1; \bar{s}_i) < \rho_N T \right\}$

Remark

Note that $\det(1; \Sigma) = Tr(\Sigma)$, *and* $\det(d; \Sigma) = \det(\Sigma)$.

Improved procedure: Jacod et al. (2008)

Main idea: check if the *difference* between the determinant of the submatrix of a given rank and the alternative rank is "big enough"

- Compute matrices of increments (2)
- Compute the statistics (3) and the "estimators" along with their variances:

$$\bar{R}_{T,\Delta}(r) = \frac{1}{\Delta^{r-1} r!} \frac{\sum_{i=1}^{N-r+1} \det(r; \bar{s}_i)}{\left(\sum_{i=1}^{N-r+1} \det(1; \bar{s}_i)\right)^r}$$

- Intuitively, if $r < r_0$, $\bar{R}_{T,\Delta}(r_0) \approx 0$
- We have *sort of* test for verifying $H_1 : r < r_0$.

Improved procedure: Jacod et al. (2008)

Limitations of the method:

- Complicated to compute when d is big
- One-sided test: two-sided test would be more convenient
- Performance for systems with non-linear drift and/or higly-degenerate diffusion matrix

When does it fail?

Example: Brownian motion

Consider the system:

$$\begin{cases} X_t^1 = \int_0^t \sigma^1 dW_t \\ X_t^2 = \int_0^t \sigma^2 dW_t \end{cases}$$

$$s_i = \begin{pmatrix} \sigma^1(W_{2i+1} - W_{2i}) & \sigma^1(W_{2i+2} - W_{2i+1}) \\ \sigma^2(W_{2i+1} - W_{2i}) & \sigma^2(W_{2i+2} - W_{2i+1}) \end{pmatrix}$$

Part III: Random perturbation approach

Going further: Jacod and Podolskij (2013)

Assume: computing $\det E$ is difficult. **But** computing $\det(E+D)$ is easy.

Then:

$$\frac{\det(E+2hD)}{\det(E+hD)} \approx 2^{d-r}$$

Because:

$$\det(E+hD) = \det E + h\gamma_{d-1}(E,D) + \cdots + h^d \det D,$$

where γ_r, $r = 1, \ldots, d$ stands for a sum of determinants of all possible matrices, whose r columns are equal to r columns of a matrix E (with the same indexes), and the remaining $d-r$ — to the corresponding columns of a matrix D.

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Determinant expansion: deterministic 3d-example

$$\det(E+hD) =$$

$$\det\begin{pmatrix} e_{11}+hd_{11} & e_{12}+hd_{12} & e_{13}+hd_{13} \\ e_{21}+hd_{21} & e_{22}+hd_{22} & e_{23}+hd_{23} \\ e_{31}+hd_{31} & e_{32}+hd_{32} & e_{33}+hd_{33} \end{pmatrix} = \det\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} +$$

$$\overbrace{h\det\begin{pmatrix} e_{11} & e_{12} & d_{13} \\ e_{21} & e_{22} & d_{23} \\ e_{31} & e_{32} & d_{33} \end{pmatrix} + h\det\begin{pmatrix} e_{11} & d_{12} & e_{13} \\ e_{21} & d_{22} & e_{23} \\ e_{31} & d_{32} & e_{33} \end{pmatrix} + h\det\begin{pmatrix} d_{11} & e_{12} & e_{13} \\ d_{21} & e_{22} & e_{23} \\ d_{31} & e_{32} & e_{33} \end{pmatrix}}^{\gamma_2(E,D)} +$$

$$\overbrace{h^2\det\begin{pmatrix} e_{11} & d_{12} & d_{13} \\ e_{21} & d_{22} & d_{23} \\ e_{31} & d_{32} & d_{33} \end{pmatrix} + h^2\det\begin{pmatrix} d_{11} & d_{12} & e_{13} \\ d_{21} & d_{22} & e_{23} \\ d_{31} & d_{32} & e_{33} \end{pmatrix} + h^2\det\begin{pmatrix} d_{11} & e_{12} & d_{13} \\ d_{21} & e_{22} & d_{23} \\ d_{31} & e_{32} & d_{33} \end{pmatrix}}^{\gamma_1(E,D)} +$$

$$h^3\det\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

Random perturbation approach

Given a d-dimensional diffusion process X, consider 2 new processes:

$$\tilde{X}_t^{(k)} = X_t + \sqrt{k\Delta}\tilde{\Sigma}\tilde{W}_t,$$

where $\tilde{\Sigma}$ is a non-random matrix of full rank.

Idea: "Measure the influence" of the perturbation by considering the statistics with different step size (here: Δ and 2Δ).

Toy example: 1-d process with constant drift and diffusion

Take the process:

$$dX_t = a dt + \sigma dW_t$$

Add the random perturbation:

$$\tilde{X}_t^{(1)} = a dt + \sigma dW_t + \sqrt{\Delta}\tilde{\sigma}\tilde{W}_t,$$
$$\tilde{X}_t^{(2)} = a dt + \sigma dW_t + \sqrt{2\Delta}\tilde{\sigma}\tilde{W}_t$$

Using the first-order approximation, compute:

$$\mathbb{E}\left[\left(\frac{\tilde{X}_{i+1}^{(k)} - \tilde{X}_i^{(k)}}{\sqrt{k\Delta}}\right)^2\right] = \sigma^2 + k\Delta a + k\Delta\tilde{\sigma} =: s_i^k$$

Notice that

$$\frac{s_i^2}{s_i^1} = \begin{cases} 1 + O(\Delta) & \text{if } \sigma \neq 0 \\ 2 & \text{if } \sigma = 0 \end{cases}$$

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Toy example: 1-d process with constant drift and diffusion

Take the process:

$$dX_t = adt + \sigma dW_t$$

Add the random perturbation:

$$\tilde{X}_t^{(1)} = adt + \sigma dW_t + \sqrt{\Delta}\tilde{\sigma}\tilde{W}_t,$$
$$\tilde{X}_t^{(2)} = adt + \sigma dW_t + \sqrt{2\Delta}\tilde{\sigma}\tilde{W}_t$$

Using the first-order approximation, compute:

$$\mathbb{E}\left[\left(\frac{\tilde{X}_{i+1}^{(k)} - \tilde{X}_i^{(k)}}{\sqrt{k\Delta}}\right)^2\right] = \sigma^2 + k\Delta a + k\Delta\tilde{\sigma} =: s_i^k$$

Notice that

$$\frac{s_i^2}{s_i^1} = \begin{cases} 1 + O(\Delta) & \text{if } \sigma \neq 0 \\ 2 & \text{if } \sigma = 0 \end{cases} \quad \Rightarrow \quad 1 - \frac{\log \mathbb{E}\frac{s_i^2}{s_i^1}}{\log 2} = \begin{cases} 1 & \text{if } \sigma \neq 0 \\ 0 & \text{if } \sigma = 0 \end{cases}$$

Random perturbation approach: key statistics

Key statistics of the test for a d-dimensional process $s_i^k,\ i = 1, 2$ is defined as follows:

$$S_T^k = 2d\Delta \sum_{i=0}^{N-1} \det \begin{pmatrix} \frac{\tilde{X}^{1,(k)}_{2id+k} - \tilde{X}^{1,(k)}_{2id}}{\sqrt{k\Delta}} & \frac{\tilde{X}^{1,(k)}_{2id+2k} - \tilde{X}^{1,(k)}_{2id+k}}{\sqrt{k\Delta}} & \dots & \frac{\tilde{X}^{1,(k)}_{2id+2d} - \tilde{X}^{1,(k)}_{2id+kd-k}}{\sqrt{k\Delta}} \\ \frac{\tilde{X}^{2,(k)}_{2id+k} - \tilde{X}^{2,(k)}_{2id}}{\sqrt{k\Delta}} & \frac{\tilde{X}^{2,(k)}_{2id+2k} - \tilde{X}^{2,(k)}_{2id+k}}{\sqrt{k\Delta}} & \dots & \frac{\tilde{X}^{2,(k)}_{2id+2d} - \tilde{X}^{2,(k)}_{2id+kd-k}}{\sqrt{k\Delta}} \\ \dots & \dots & \dots & \dots \\ \frac{\tilde{X}^{d,(k)}_{2id+k} - \tilde{X}^{d,(k)}_{2id}}{\sqrt{k\Delta}} & \frac{\tilde{X}^{d,(k)}_{2id+2k} - \tilde{X}^{d,(k)}_{2id+k}}{\sqrt{k\Delta}} & \dots & \frac{\tilde{X}^{d,(k)}_{2id+2d} - \tilde{X}^{d,(k)}_{2id+kd-k}}{\sqrt{k\Delta}} \end{pmatrix}^2$$

Random perturbation approach: estimator

Define:

$$\hat{R}_{T,\Delta} = d - \frac{\log \frac{S_T^2}{S_T^1}}{\log 2}$$

$$V(T,\Delta) := Var\left[\widehat{R}_{T,\Delta}\right] = \frac{\left(\frac{E[S_T^1]}{E[S_T^2]}\right)^2 Var[S_T^2] - 2\frac{E[S_T^1]}{E[S_T^2]} Cov[S_T^1 S_T^2] + Var[S_T^1]}{(E[S_T^1]\log 2)^2}.$$

Corollary (3.6 in Jacod and Podolskij (2013))

$$\frac{\widehat{R}(T,\Delta) - r_0}{\sqrt{\Delta V(T,\Delta)}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

Random perturbation approach: testing procedure

Summary:

$$H_0 : rank(\Sigma) = r_0$$
$$H_1 : rank(\Sigma) \neq r_0$$

Decision rule: reject H_0 with $(1-\alpha)\%$ confidence level if

$$q_{\frac{\alpha}{2}} \nleq \frac{\widehat{R}(T,\Delta) - r_0}{\sqrt{\Delta V(T,\Delta)}} \nleq q_{1-\frac{\alpha}{2}}$$

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Problem: choose $\tilde{\Sigma}$ wisely!

Question 1: why do we need perturbations?

Example: Brownian motion

Consider the system:

$$\begin{cases} X_t^1 = \int_0^t \sigma^1 dW_t \\ X_t^2 = \int_0^t \sigma^2 dW_t \end{cases}$$

$$s_i^k = \frac{1}{k\Delta} \det \begin{pmatrix} \sigma^1(W_{2i+k} - W_{2i}) & \sigma^1(W_{2i+2k} - W_{2i+k}) \\ \sigma^2(W_{2i+k} - W_{2i}) & \sigma^2(W_{2i+2k} - W_{2i+k}) \end{pmatrix}^2$$

Note that $\mathbb{E}[s_i^k] = 0$ for $k = 1, 2$.

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Question 1: why do we need perturbations?

Example: Brownian motion

Consider the system:

$$\begin{cases} \tilde{X}_t^{1,(k)} = \int_0^t \sigma^1 dW_t + \sqrt{k\Delta}\tilde{\sigma} d\tilde{W}_t^1 \\ \tilde{X}_t^{2,(k)} = \int_0^t \sigma^2 dW_t + \sqrt{k\Delta}\tilde{\sigma} d\tilde{W}_t^2 \end{cases}$$

$$s_i^k = \frac{1}{k\Delta} \det \begin{pmatrix} \sigma^1(W_{2i+k} - W_{2i}) + \sqrt{k\Delta}\tilde{\sigma}(\tilde{W}^1_{2i+k} - \tilde{W}^1_{2i}) & \sigma^1(W_{2i+2k} - W_{2i+k}) + \sqrt{k\Delta}\tilde{\sigma}(\tilde{W}^1_{2i+2k} - \tilde{W}^1_{2i+k}) \\ \sigma^2(W_{2i+k} - W_{2i}) + \sqrt{k\Delta}\tilde{\sigma}(\tilde{W}^2_{2i+k} - \tilde{W}^2_{2i}) & \sigma^2(W_{2i+2k} - W_{2i+k}) + \sqrt{k\Delta}\tilde{\sigma}(\tilde{W}^2_{2i+2k} - \tilde{W}^2_{2i+k}) \end{pmatrix}^2$$

Note that $\mathbb{E}[s_i^k] = k\Delta(\tilde{\sigma})^2$ for $k = 1, 2$.

Question 2: how to choose $\tilde{\Sigma}$?

Ongoing work

A. M., Adeline Samson, Patricia Reynaud-Bouret

Assume $\tilde{\Sigma} = \tilde{\sigma} I$. The increments of the perturbed process can be decomposed as:

$$\frac{\tilde{X}_{i+k}^{j,(k)} - \tilde{X}_i^{j,(k)}}{\sqrt{k\Delta}} \approx \frac{1}{\sqrt{k\Delta}} \sum_{l=1}^{q} \sigma_i^{jl} \left(W_{i+k}^{jl} - W_i^{jl} \right) + \sqrt{k\Delta} \left(a_i^j + \frac{1}{\sqrt{k\Delta}} \tilde{\sigma} \left(\tilde{W}_{i+k}^j - \tilde{W}_i^j \right) \right) + \frac{1}{\sqrt{k\Delta}} a_i'^j \sum_{l=1}^{q} \int_{i\Delta}^{(i+k)\Delta} \sigma_i^{jl} \left(W_{i+k}^{jl} - W_i^{jl} \right) dW_s^j + O(\Delta^{\frac{3}{2}})$$

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Question 2: how to choose $\tilde{\Sigma}$?

Ongoing work

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Assume $\tilde{\Sigma} = \tilde{\sigma} I$. The increments of the perturbed process can be decomposed as:

$$\frac{\tilde{X}^{j,(k)}_{i+k} - \tilde{X}^{j,(k)}_{i}}{\sqrt{k\Delta}} \approx \overbrace{\frac{1}{\sqrt{k\Delta}} \sum_{l=1}^{q} \sigma^{jl}_i \left(W^{jl}_{i+k} - W^{jl}_i \right)}^{\sum_{l=1}^{q} \sigma^{jl}_i \cdot \mathcal{N}(0,1)} +$$

$$\sqrt{k\Delta} \overbrace{\left(a^j_i + \frac{1}{\sqrt{k\Delta}} \tilde{\sigma} \left(\tilde{W}^j_{i+k} - \tilde{W}^j_i \right) \right)}^{a^j_i + \tilde{\sigma} \cdot \mathcal{N}(0,1)} +$$

$$\frac{1}{\sqrt{k\Delta}} a'^j_i \sum_{l=1}^{q} \int_{i\Delta}^{(i+k)\Delta} \sigma^{jl}_i \left(W^{jl}_{i+k} - W^{jl}_i \right) dW^j_s + O(\Delta^{\frac{3}{2}})$$

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Part IV: Numerical experiments

General setting

1. Generate 1000 trajectories with $\Delta = 1e - 5$, $T = 10$, using 1.5 strong order scheme
2. Subsample the data with a bigger Δ (see tables for details)
3. Compute test statistics $\hat{R}(T, \Delta)$ and $V(T, \Delta)$
4. Test the "true" and a "wrong" hypothesis
5. Report the results

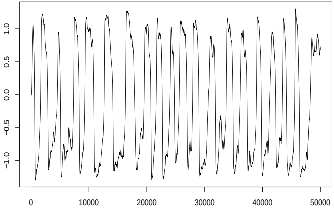
Example 1: FitzHugh-Nagumo model

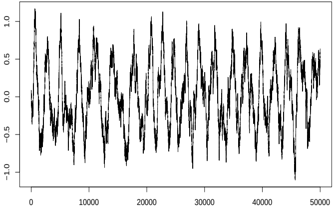
The behaviour of the neuron is defined through:

$$\begin{cases} dX_t = \frac{1}{\varepsilon}(X_t - X_t^3 - Y_t - s)dt + \sigma_1 dW_t^1 \\ dY_t = (\gamma X_t - Y_t + \beta)dt + \sigma_2 dW_t^2 \end{cases}$$

- X_t — membrane potential
- Y_t — recovery variable
- s — magnitude of the stimulus current

Parameters used in simulations:
$\varepsilon = 0.1, \beta = 0.3, \gamma = 1.5, s = 0.01.$





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Numerical performance: FitzHugh-Nagumo model

$r_0 = 2$	$\tilde{\sigma} = 0$	$\tilde{\sigma} = 0.5$	$\tilde{\sigma} = 1$	$\tilde{\sigma} = 10$
$\Delta = 0.1$	0.653 (0.964) 47.9%, 80.9%	0.655 (0.966) 48%, 20.4%	0.655 (0.966) 49.7%, 19.5%	0.056 (0.805) 78.6%, 38.1%
$\Delta = 1e-2$	1.713 (0.293) 20.7%, 99%	1.699 (0.292) 22.4%, 42.8%	1.656 (0.286) 26.9%, 67.9%	0.407 (0.254) 100%, 65.4%
$\Delta = 1e-3$	1.962 (0.082) 8.8%, 100%	1.960 (0.082) 9.4%, 100%	1.954 (0.082) 10%, 100%	1.410 (0.085) 100%, 99.8%
$\Delta = 1e-4$	1.998 (0.026) 6.1%, 100%	1.995 (0.026) 6.2%, 100%	1.995 (0.026) 6.1%, 100%	1.920 (0.026) 87.4%, 100%

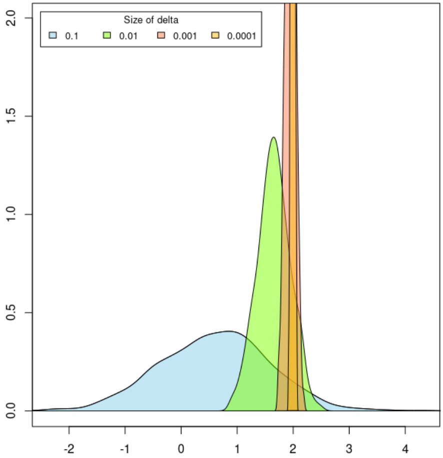
Table: Results of the 5%-test for FitzHugh-Nagumo: **elliptic case**. First line: mean value of $\hat{R}(\Delta, T)$ and its standard deviation, second line: percent of rejections of the true H_0 hypothesis and false H_0 ($r = r_0 - 1$)

Numerical performance: FitzHugh-Nagumo model

$r_0 = 1$	$\tilde{\sigma} = 0$	$\tilde{\sigma} = 0.5$	$\tilde{\sigma} = 1$	$\tilde{\sigma} = 10$
$\Delta = 0.1$	0.436 (1.044) 22.8%, 52.3%	0.432 (1.032) 23.1%, 55.7%	0.452 (1.040) 25.7%, 58.6%	0.044 (0.807) 38.1%, 12.7%
$\Delta = 1e-2$	0.950 (0.478) 9.6%, 68.2%	0.942 (0.470) 11.2%, 92%	0.913 (0.435) 10.3%, 98.1%	0.204 (0.266) 87%, 14.8%
$\Delta = 1e-3$	0.994 (0.147) 4.8%, 100%	0.993 (0.145) 4.9%, 100%	0.989 (0.138) 5.6%, 100%	0.711 (0.085) 93.5%, 100%
$\Delta = 1e-4$	0.997 (0.047) 5.2%, 100%	0.997 (0.026) 4.7%, 100%	0.997 (0.047) 4.8%, 100%	0.961 (0.027) 30.9%, 100%

Table: Results of the 5%-test for FitzHugh-Nagumo: **hypoelliptic** case. First line: mean value of $\hat{R}(\Delta, T)$ and its standard deviation, second line: percent of rejections of the true H_0 hypothesis and false H_0 ($r = r_0 - 1$)

Numerical performance: FitzHugh-Nagumo model



Example 2: diffusion aproximation of Hawkes process

Approximating SDE describing the spiking activity in a network of neurons, consisting of 2 subpopulations:

$$dZ_t = A(Z_t)dt + B(Z_t)dW_t.$$

- 2 populations: **inhibitory** and **excitatory** (2 *rough* variables)
- 2 or 4 memory variables in each populations (respectively, 4 or 10-dimensional system)
- 100 neurons in each population

Numerical performance: Diffusion approximation of Hawkes process

$r_0 = 2$	$\tilde{\sigma} = 0$	$\tilde{\sigma} = 0.1$	$\tilde{\sigma} = 0.5$	$\tilde{\sigma} = 1$
$\Delta = 0.1$		-1.177 (2.398) 70.1%, 59.3%	0.338 (2.678) 62%, 58.4%	0.308 (2.195) 56.1%, 42.6%
$\Delta = 1e-2$		1.543 (1.157) 21.6%, 20.8%	1.043 (0.979) 34.3%, 14.5%	0.850 (0.831) 48.1%, 13.2%
$\Delta = 1e-3$		2.032 (0.386) 4.1%, 73.3%	1.782 (0.369) 14.2%, 57.9%	1.179 (0.291) 79.9%, 8.5%
$\Delta = 1e-4$		2.059 (0.137) 4.2%, 100%	1.903 (0.143) 9.6%, 100%	1.805 (0.132) 31.8%, 99.9%

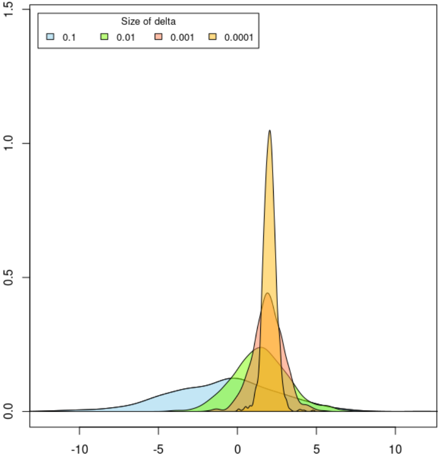
Table: Results of the 5%-test for diffusion approximation model ($d = 4$). First line: mean value of $\hat{R}(\Delta, T)$ and its standard deviation, second line: percent of rejections of the true H_0 hypothesis and false H_0 ($r = r_0 - 1$)

Numerical performance: Diffusion approximation of Hawkes process

$r_0 = 2$	$\tilde{\sigma} = 0$	$\tilde{\sigma} = 0.1$	$\tilde{\sigma} = 0.5$	$\tilde{\sigma} = 1$
$\Delta = 0.1$	😵	-1.023 (3.856) 75.9%, 72.1%	-0.101 (3.249) 60.1%, 57%	0.300 (3.199) 57.9%, 53.4%
$\Delta = 1e - 2$	😵	1.211 (1.784) 33.2%, 27.9%	1.161 (1.471) 30.6%, 23.3%	1.069 (1.450) 33.6%, 23.6%
$\Delta = 1e - 3$	😵	1.822 (0.889) 15.2%, 34.3%	1.739 (0.772) 13.3%, 29.9%	1.347 (0.655) 31.5%, 15.6%
$\Delta = 1e - 4$	😵	2.004 (0.359) 6.9%, 84.8%	1.975 (0.348) 5.8%, 84.6%	1.94 (0.336) 7.9%, 87.9%

Table: Results of the 5%-test for diffusion approximation model ($d = 10$). First line: mean value of $\hat{R}(\Delta, T)$ and its standard deviation, second line: percent of rejections of the true H_0 hypothesis and false H_0 ($r = r_0 - 1$)

Numerical performance: Diffusion approximation of Hawkes process



General comments

- For **deterministic systems**: in general, $\tilde{\sigma}$ has no influence on accuracy, BUT setting $\tilde{\sigma} = 0$ breaks the test.
- For **elliptic systems**: perturbations diminish the accuracy, setting $\tilde{\sigma} = 0$ seems to be the optimal choice.
- For **hypoelliptic systems**: highly-degenerate systems are sensitive to $\tilde{\sigma}$. If $\tilde{\sigma}$ is too big, we will not learn that the system is hypoelliptic, if it's too small, the algorithm will break!

Possible solutions:

- Bound the $\tilde{\sigma}$ to the empirical variance. It is observable, but we need to go back to thresholds.
- Work with a sequence of $\tilde{\sigma}_N$, bounded to the variance?

Literature I

Ableidinger, M., Buckwar, E., and Hinterleitner, H. (2017). A stochastic version of the jansen and rit neural mass model: Analysis and numerics. *The Journal of Mathematical Neuroscience*, 7(1):1.

Buckwar, E., Tamborrino, M., and Tubikanec, I. (2). Spectral density-based and measure-preserving abc for partially observed diffusion processes. an illustration on hamiltonian sdes.

Chevallier, J. (2017). Mean-field limit of generalized Hawkes processes. *Stochastic Processes and their Applications*, 127(12).

Ditlevsen, S. and Löcherbach, E. (2017). Multi-class oscillating systems of interacting neurons. *SPA*, 127:1840–1869.

Fitzhugh, R. (1961). Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal*, 1(6):445–466.

Hodgkin, A. L. and Huxley, A. F. (1952). A quantitative description of membrane currents and its application to conduction and excitation in nerve. *Journal of Physiology-London*, 117(4):500–544.

Iooss, B. and Lemaître, P. (2015). A review on global sensitivity analysis methods. In Meloni, C. and Dellino, G., editors, *Uncertainty management in Simulation-Optimization of Complex Systems: Algorithms and Applications*. Springer.

Jacod, J., Lejay, A., Talay, D., et al. (2008). Estimation of the brownian dimension of a continuous itô process. *Bernoulli*, 14(2):469–498.

Jacod, J. and Podolskij, M. (2013). A test for the rank of the volatility process: the random perturbation approach. *The Annals of Statistics*, 41(5):2391–2427.

Literature II

Jansen, B. H. and Rit, V. G. (1995). Electroencephalogram and visual evoked potential generation in a mathematical model of coupled cortical columns. *Biological Cybernetics*, 73(4):357.

Konstantinides, K. and Yao, K. (1988). Statistical analysis of effective singular values in matrix rank determination. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 36(5):757–763.

Kritchman, S. and Nadler, B. (2008). Determining the number of components in a factor model from limited noisy data. *Chemometrics and Intelligent Laboratory Systems*, 94(1):19–32.

Morris, C. and Lecar, H. (1981). Voltage oscillations in the barnacle giant muscle fiber. *Biophysical journal*, 35(1):193–213.

Tuckwell, H. C. (2005). *Introduction to theoretical neurobiology: volume 2, nonlinear and stochastic theories*, volume 8. Cambridge University Press.

Zarowski, C. J. (1998). The mdl criterion for rank determination via effective singular values. *IEEE transactions on signal processing*, 46(6):1741–1744.